PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 13, Fasc. 2 (1992), pp. 321–343

COMPLETENESS IN LOCATION FAMILIES

BY

M. ISENBECK AND L. RÜSCHENDORF (MÜNSTER)

Abstract. Some statistical completeness results are proved for location families. For the proofs in the first step some generalizations of the Wiener closure theorem to L^p-spaces with weight functions are established. The idea then is to relate the statistical completeness notions to the functional analytic notions of completeness in those weighted L^p-spaces.

1. INTRODUCTION

Let $P = f\lambda^1$ be a probability measure on $(\boldsymbol{R}^1, \mathscr{B}^1)$ with Lebesgue density f and let $\mathscr{F} := \{f_\theta(x) = f(x-\theta)\colon \theta \in \boldsymbol{R}^1\}$ denote the densities of the location family $\mathscr{P}$ generated by P. The Wiener closure theorem says that $\mathscr{P}$ is boundedly complete if and only if $\hat{f}(t) \neq 0$ for all $t \in \boldsymbol{R}^1$, where $\hat{f}(t) = \int e^{-itx} f(x)\,dx$ denotes the Fourier transform of f. This theorem extends to general locally compact abelian groups, but we will restrict in this paper to the case of the real line. The Wiener closure theorem has been used in the context of estimation theory in papers [10] and [6]. Stronger forms of the completeness of location families have been established in the literature only in very exceptional cases as, e.g., for normal translation families, using the well-known completeness result on exponential families. Some related completeness results based on analyticity properties of characteristic functions can be found in [5], [16], and [32]. Some further examples can be found in [23].

For $1 \leqslant q \leqslant \infty$ let

$$\mathscr{L}^q = \mathscr{L}^q(\lambda^1) = \{f\colon (\boldsymbol{R}^1, \mathscr{B}^1) \to (\boldsymbol{R}^1, \mathscr{B}^1);\ \int |f(x)|^q dx < \infty\} \tag{1}$$

and

$$\mathscr{L}^q(\mathscr{F}) = \{g\colon (\boldsymbol{R}^1, \mathscr{B}^1) \to (\boldsymbol{R}^1, \mathscr{B}^1);\ \int |g(x)|^q f(x-\theta)\,dx < \infty,\quad \forall \theta \in \boldsymbol{R}^1\}. \tag{2}$$

If $\mathscr{F} \subset \mathscr{L}^p$, where $1/p + 1/q = 1$, then

(3) $\mathscr{F}$ is called *$\mathscr{L}^q$-complete* if $g \in \mathscr{L}^q$ and

$$g * f^*(\theta) = \int g(x) f(x-\theta)\,dx = 0,\ \forall \theta,\ \text{implies that } g = 0\ [\lambda^1],$$

where $f^*(x) := f(-x)$. In a statistical context there is a stronger notion of completeness:

(4) $\mathscr{F}$ is *$\mathscr{L}^q(\mathscr{F})$-complete* if $g \in \mathscr{L}^q(\mathscr{F})$ and

$$\int g(x) f(x-\theta)\,dx = 0,\ \forall \theta,\ \text{implies that } g = 0\ [\lambda^1].$$

The $\mathscr{L}^\infty = \mathscr{L}^\infty(\mathscr{F})$-completeness is also called *bounded completeness of $\mathscr{F}$*, while the $\mathscr{L}^1(\mathscr{F})$-completeness is called *completeness of $\mathscr{F}$* in the statistical literature. Since $\mathscr{L}^\infty \subset \mathscr{L}^q(\mathscr{F})$ for $1 \leqslant q < \infty$, the condition

(5) $$Z\hat{f} := \{t\colon \hat{f}(t) = 0\} = \emptyset$$

is a necessary condition for $\mathscr{L}^q(\mathscr{F})$-completeness.

The idea of this paper is to establish extensions of the Wiener closure theorem to weighted $\mathscr{L}^q$-spaces (so-called Beurling algebras) in order to derive from these results necessary and/or sufficient conditions for the $\mathscr{L}^q(\mathscr{F})$-completeness of $\mathscr{F}$. Since the literature on the generalization of the Wiener closure theorem is somewhat scattered and abstract and not easy accessible to a non-specialist in harmonic analysis, we also include for the ease of reference some results known in the literature.

To see the difference between the notions of $\mathscr{L}^q$- and $\mathscr{L}^q(\mathscr{F})$-completeness take, e.g., the Cauchy density

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Then for $g \in \mathscr{L}^1$, $\theta \in \boldsymbol{R}^1$,

$$H_g(\theta) := f^* * g(\theta) = \frac{1}{\pi}\int \frac{1}{1+(x-\theta)^2}\, g(x)\,dx$$

is called the *Poisson transformation* of g. If $f^* * g \equiv 0$, then $\hat{f}^* \hat{g} = 0$ implying that $\hat{g} = 0$, since $Z\hat{f} = \emptyset$ and, therefore, $g = 0$. This result is equivalent to the $\mathscr{L}^1$-completeness of $\mathscr{F} = \{L_\theta f;\ \theta \in \boldsymbol{R}^1\}$, $L_\theta f(x) := f(x-\theta)$. The more involved $\mathscr{L}^1(\mathscr{F})$-completeness was established in [25] by proving a general inversion formula for the Poisson transformation. It was used in statistical context in [10] and [23]. Our methods allow us to establish the $\mathscr{L}^q(\mathscr{F})$-completeness for $q > 2$ only (cf. Section 4).

The functional analytic completeness results for $\mathscr{L}^q$ are also of relevance in estimation theory but allow to deal with a more restricted class of estimators. For this reason it seems to be justified to give also a fairly detailed representation of these results. A more complete exposition of these results is given in [15].

2. WIENER'S THEOREM FOR BEURLING ALGEBRAS

A measurable function w: $\boldsymbol{R}^1 \to \boldsymbol{R}^1$ is called a *weight function* if
(a) $w(x) \geqslant 1$,
(b) $w(x+y) \leqslant w(x)w(y)$,
(c) $w(x/\varrho) \leqslant w(x)$, $\forall \varrho \geqslant 1$, $x, y \in \boldsymbol{R}^1$.
For a weight function w define

(6)
$$\mathscr{L}^p_w := \{f\colon \boldsymbol{R}^1 \to \boldsymbol{R}^1;\ \|f\|_{p,w} := \|fw\|_p < \infty\},$$
$$\mathscr{L}^q_{w^-} := \{f\colon \boldsymbol{R}^1 \to \boldsymbol{R}^1;\ \|f\|_{q,w^-} := \|fw^{-1}\|_q < \infty\},$$

where $\|f\|_q$ is the q-norm in $\mathscr{L}^q(\lambda^1)$. Then $(\mathscr{L}^p_w)' = \mathscr{L}^q_{w^-}$ for $1 \leqslant p < \infty$, $1/p+1/q = 1$ and for $H \subset \mathscr{L}^p_w$ the following holds (by a Hahn–Banach argument):

(7) H is $\mathscr{L}^q_{w^-}$-complete if and only if $\operatorname{lin} H$ is dense in $\mathscr{L}^p_w$.

For the general theory of the Beurling algebras $\mathscr{L}^p_w$ we refer to the books of Reiter [27], Benedetto [1], Hewitt and Ross [13], and Donoghue [9], and to the articles [3], [8], and [31].

2.1. The case $p = 1$. Let us put

(8)
$$\mathscr{A}^1_w := \{\hat{f}(t)\colon f \in \mathscr{L}^1_w\} \quad \text{with } \|\hat{f}\|_{\mathscr{A}^1_w} := \|f\|_{1,w},$$
$$\mathscr{K}_w(E) := \{\hat{f} \in \mathscr{A}^1_w\colon E \subset Z\hat{f}\} \quad \text{for } E \subset \boldsymbol{R}^1 \text{ closed and}$$
$$\mathscr{A}^1_w(E) := \mathscr{A}^1_w | \mathscr{K}_w(E).$$

$\mathscr{K}_w(E)$ is a closed ideal in the commutative Banach algebra $\mathscr{A}^1_w$, which is isomorphic to $\mathscr{L}^1_w$, and $\mathscr{A}^1_w(E)$ is a commutative Banach algebra with unit (in the quotient topology). There exists an element $\varphi \in \mathscr{A}^1_w$ with compact support, such that $\varphi(t) = 1$ for all $t \in E$. The projection $p(\varphi)$ of φ to $\mathscr{A}^1_w(E)$ is a unit of $\mathscr{A}^1_w(E)$. If $\hat{f} \in \mathscr{A}^1_w(E)$ with $E \cap Z(\hat{f}) = \emptyset$, then a well-known theorem on Banach algebras implies that $1/p(\hat{f}) \in \mathscr{A}^1_w(E)$, and so there exists an element $\hat{g} \in \mathscr{A}^1_w$ with

(9)
$$\hat{g}(t) = 1/\hat{f}(t) \quad \text{for } t \in E.$$

This means that the Wiener–Lévy theorem holds for $\mathscr{L}^1_w$ (for a different proof cf. [27], p. 16). Wiener's theorem for $\mathscr{L}^1_w$ now has the following form (cf. [27], p. 16). We give a proof of this result since we shall use some of the arguments of this proof in the following part. By $\operatorname{lin}\mathscr{F}$ we denote the linear hull of $\mathscr{F}$.

THEOREM 1. *If* $f \in \mathscr{L}^1_w$ *and* $\mathscr{F} = \{L_\theta f;\ \theta \in \boldsymbol{R}^1\}$, *then:*
$\operatorname{lin}\mathscr{F}$ *is dense in* $\mathscr{L}^1_w \Leftrightarrow Z\hat{f} = \emptyset \Leftrightarrow \mathscr{F}$ *is* $\mathscr{L}^1_{w^-}$*-complete.*

Proof. ($\Rightarrow$) If $\hat{f}(t_0)=0$ and $g\in\mathscr{L}^1_w$ with $|\hat{g}(t_0)|=\varepsilon>0$, then

$$\begin{aligned}\|g-\textstyle\sum c_i L_i f\|_{1,w} &= \int |g(t)-\textstyle\sum c_i f(t-y_i)|\, w(t)\,dt\\ &\geqslant \left|\int (g(t)-\textstyle\sum c_i f(t-y_i))e^{itt_0}dt\right| = \left|\hat{g}(t_0)-(\textstyle\sum c_i e^{-iy_it_0})\hat{f}(t_0)\right|\\ &= |\hat{g}(t_0)| = \varepsilon > 0\end{aligned}$$

in contradiction to the assumption.

($\Leftarrow$) For the converse direction let $g\in\mathscr{L}^1_w$. Then

1. *$\exists g_0\in\mathscr{L}^1_w$ with* $\operatorname{supp}(\hat{g}_0)$ *compact such that* $\|g-g_0\|_{1,w}<\varepsilon$.

This follows by a standard technique.

2. *$\exists h\in\mathscr{L}^1_w$ with $h*f=g_0$.*

Since $Z\hat{f}\cap\operatorname{supp}(\hat{g}_0)=\emptyset$, by (9) there is an element $f_1\in\mathscr{L}^1_w$ with $\hat{f}_1(t)=1/\hat{f}(t)$ for $t\in\operatorname{supp}(\hat{g}_0)$. With $h:=g_0*f_1$ the following holds: $h\in\mathscr{L}^1_w$ and

$$\begin{aligned}(h*f)^\wedge(t)=\hat{g}_0(t)\hat{f}_1(t)\hat{f}(t) &= \begin{cases}0=\hat{g}_0(t), & t\notin\operatorname{supp}(\hat{g}_0),\\ \hat{g}_0(t), & t\in\operatorname{supp}(\hat{g}_0),\end{cases}\\ &= \hat{g}_0(t).\end{aligned}$$

3. *$\exists h_0\in\mathscr{L}^1_w$ with* $\operatorname{supp}(h_0)$ *compact and* $\|h*f-h_0*f\|_{1,w}<\varepsilon$.

(Proof as in step 1.)

4. *$\exists y_i\in\boldsymbol{R}$, $\lambda_i\in\boldsymbol{R}$, $n\in\boldsymbol{N}$, with* $\left\|h_0*f-\sum\lambda_i L_{y_i}f\right\|_{1,w}<\varepsilon$.

The translation operator L_y is continuous w.r.t. the $\|\ \|_{1,w}$-norm. Therefore, for $|y|<\delta$ we have

$$\|L_yf-f\|_{1,w}<\frac{2}{\|h_0\|_1}M^{-1},\qquad M:=\sup\{w(x)\colon x\in\operatorname{supp}(h_0)\}.$$

$\exists y_1,\ldots,y_n\in\operatorname{supp}(h_0)$ such that $\operatorname{supp}(h_0)\subset\bigcup_{j=1}^n U_\delta(y_j)$, so

$$\operatorname{supp}(h_0)=\bigcup_{i=1}^n A_i,$$

(A_i) being the disjoint union of the $U_\delta(y_i)$. Therefore,

$$\begin{aligned}&\left\|h_0*f-\textstyle\sum\lambda_i L_{y_i}f\right\|_{1,w}\\ &\quad\leqslant \sum_i \left|\int_{A_i}|h_0(x)|\,|L_xf(y)-L_{y_i}f(y)|\,dx\right| w(y)\,dy\\ &\quad= \sum_i \int_{A_i}|h_0(x)|\,\|L_xf-L_{y_i}f\|_{1,w}\,dx\\ &\quad\leqslant M\sum_{i=1}^n\int_{A_i}|h_0(x)|\,\|f-L_{y_i-x}f\|_{1,w}\,dx<\varepsilon,\qquad \text{where } \lambda_i=\int_{A_i}h_0(x)\,dx.\end{aligned}$$

Steps 1–4 imply the result by the triangle inequality. ∎

More generally, for subsets $\mathscr{I} \subset \mathscr{L}_w^1$ the following holds true:

(10) $\mathscr{I}$ is a closed translation invariant subspace of $\mathscr{L}_w^1$ iff $\hat{\mathscr{I}} = \{\hat{f}; f \in \mathscr{I}\}$ is a closed ideal in $\mathscr{A}_w^1$.

The *synthesis problem* is the question whether

$$\mathscr{I} = \{h \in \mathscr{L}_w^1 \colon \hat{h}(t) = 0,\ \forall t \in Z(\hat{\mathscr{I}}) := \bigcap_{f \in \mathscr{I}} Z\hat{f}\}. \tag{11}$$

The inclusion "$\subset$" is trivially satisfied and is called the *Tauberian condition*; the inclusion "$\supset$" is called the *synthesis condition.*

A subset $P \subset \boldsymbol{R}$ is called *perfect* if $P = \bar{P}$ and if P has no isolated points. $S \subset \boldsymbol{R}$ is called *scattered* if S does not contain a perfect subset. Consider the weight function $w_\alpha(x) := (1+|x|)^\alpha$. Then the following weakened synthesis condition holds.

THEOREM 2. *For* $\alpha \in [0, 1)$ *and a closed ideal* $\hat{\mathscr{I}} \subset \mathscr{A}_{w_\alpha}^1$ *the following holds*:
(a) $\hat{\mathscr{I}} \subset \{\hat{h} \in \mathscr{A}_{w_\alpha}^1 \colon \hat{h}(t) = 0,\ \forall t \in Z(\mathscr{I})\}$;
(b) $\hat{\mathscr{I}} \supset \{\hat{h} \in \mathscr{A}_{w_\alpha}^1 \colon \hat{h}(t) = 0,\ \forall t \in Z(\hat{\mathscr{I}})$, *such that* $\partial(Z\hat{h}) \cap \partial(Z(\hat{\mathscr{I}}))$ *is scattered*$\}$.

Proof (cf. [27], pp. 28, 132, 133). The proof uses the following lemmas:

LEMMA 1. *If* $\hat{\mathscr{I}} \subset \mathscr{A}_w^1$ *is a closed ideal*, $f \in \mathscr{L}_w^1$, w *is a weight function, and* $x \notin Z(\hat{\mathscr{I}})$, *then there exists a neighbourhood* $U_\varepsilon(x)$ *and* $\hat{h} \in \hat{\mathscr{I}}$ *with* $\hat{h}(y) = \hat{f}(y)$, $\forall y \in U_\varepsilon(x)$.

LEMMA 2. *If* $f \in \mathscr{L}_{w_\alpha}^1$ *and* $\hat{f}(x_0) = 0$, *then* $\exists (h_n) \subset \mathscr{L}_{w_\alpha}^1$, $\varepsilon_n > 0$, *such that*

$$\hat{h}_n(y) = 1,\ \forall y \in U_{\varepsilon_n}(x_0),\ \lim \|\hat{f}\hat{h}_n\|_{\mathscr{A}_{w_\alpha}^1} = 0.$$

This lemma needs the special weight function w_α.

LEMMA 3 (location lemma). *If* $\hat{\mathscr{I}} \subset \mathscr{A}_w^1$ *is a closed ideal*, $\hat{f} \in \mathscr{A}_w^1$ *belonging locally to* $\hat{\mathscr{I}}$ (*i.e.*, $\forall x \in \boldsymbol{R}^1 \colon \exists \varepsilon_x > 0 \colon \exists \hat{h}_x \in \hat{\mathscr{I}}$ *with* $\hat{h}_x(y) = \hat{f}(y)$, $\forall y \in U_{\varepsilon_x}(x)$), *then* $\hat{f} \in \hat{\mathscr{I}}$.

Proof. Since $\hat{\mathscr{I}} \subset \mathscr{A}_w^1$ is closed and $C_K \subset \mathscr{L}_w^1$ is dense, it is enough to consider f with compact support. There exist $x_1, \ldots, x_n \in \operatorname{supp}(\hat{f})$ such that

$$\operatorname{supp}(\hat{f}) \subset \bigcup_{i=1}^{n} U_{\varepsilon_{x_i}}(x_i).$$

Furthermore, there exist $k_i \in \mathscr{L}_w^1$ with

$$\hat{k}_i(y) = \begin{cases} 1 & \text{for } |x-y| < \varepsilon_{x_i}/2, \\ 0 & \text{elsewhere}. \end{cases}$$

Defining

$$\hat{l}_1 := \hat{k}_1, \quad \hat{l}_i := \hat{k}_i \prod_{j=1}^{i-1} (1-\hat{k}_j),$$

we obtain $\hat{k}_1 \in \mathscr{A}^1_w$ and $\hat{l}_i \in \mathscr{A}^1_w$ and it is easy to verify that

$$\hat{f} = \sum_{i=1}^{n} \hat{l}_i \hat{h}_{x_i}, \quad \text{i.e.,} \quad \hat{f} \in \hat{\mathscr{I}}. \ \blacksquare$$

LEMMA 4. *If* $\hat{\mathscr{I}} \subset \mathscr{A}^1_w$ *is a closed ideal,* $\hat{f} \in \mathscr{A}^1_w$, $Z(\hat{\mathscr{I}}) \subset Z\hat{f}$, *and*

$$P(\hat{f}, \hat{\mathscr{I}}) := \{x \in \mathbf{R}^1 \colon \forall \varepsilon_x > 0,\ \forall \hat{h}_x \in \hat{\mathscr{I}},\ \exists y \in U_{\varepsilon_x}(x) \colon \hat{h}_x(y) \neq \hat{f}(y)\},$$

then $P(\hat{f}, \hat{\mathscr{I}}) \cap \partial(Z\hat{f}) \cap \partial(Z\hat{\mathscr{I}})$ *is a perfect set.*

For the proof cf. [27], p. 28.

Now the proof of Theorem 2 follows from the following steps:

1. $\forall x \in (Z\hat{f})^\circ \supset (Z\hat{\mathscr{I}})^\circ$, $\exists \varepsilon_x > 0 \colon \forall y \in U_{\varepsilon_x}(x)$, $\hat{f}(y) = 0$; so $\hat{f}$ belongs locally to $\hat{\mathscr{I}}$ and, therefore, by Lemma 3, $\hat{f} \in \hat{\mathscr{I}}$.

2. $\forall x \in (Z\hat{\mathscr{I}})^c$ by Lemma 1 there exists $\varepsilon_x > 0$, $\hat{h}_x \in \hat{\mathscr{I}}$, such that

$$\forall y \in U_{\varepsilon_x}(x),\ \hat{h}_x(y) = \hat{f}(y).$$

3. By 1 and 2 and Lemma 4, $P(\hat{f}, \hat{\mathscr{I}})$ is a perfect subset of $\partial(Z\hat{f}) \cap \partial(Z\hat{\mathscr{I}})$. Since, by our assumption, $\partial(Z\hat{f}) \cap \partial(Z\hat{\mathscr{I}})$ is scattered, we conclude that $P(\hat{f}, \hat{\mathscr{I}}) = \emptyset$ and, therefore, by the localization lemma, $f \in \mathscr{I}$. ■

The special weight function w_α is only used in Lemma 2. Therefore, as a consequence of the proof of Theorem 2, for any weight function w we obtain

THEOREM 3. *If* $\hat{\mathscr{I}} \subset \mathscr{A}^1_w$ *is a closed ideal, then* $\hat{\mathscr{I}} \supset \{\hat{h} \in \mathscr{A}^1_w \colon (Z\hat{h})^\circ \supset \supset Z\hat{\mathscr{I}}\}$. ■

A corollary to this theorem is the generalized Wiener theorem:

THEOREM 4. (a) *If* $\mathscr{I} \subset \mathscr{L}^1_w$ *is closed and translation invariant, then*

$$Z\hat{\mathscr{I}} = \emptyset \Leftrightarrow \mathscr{I} = \mathscr{L}^1_w.$$

(b) *If* $(f_j)_{j \in J} \subset \mathscr{L}^1_w$ *and* $\mathscr{F} = \{L_y f_j \colon j \in J,\ y \in \mathbf{R}^1\}$, *then*:

$$\bigcap_{j \in J} Z\hat{f}_j = \emptyset \Leftrightarrow \mathscr{F} \text{ is } \mathscr{L}^\infty_w\text{-complete} \Leftrightarrow \operatorname{lin} \mathscr{F} \text{ is dense in } \mathscr{L}^1_w. \ \blacksquare$$

2.2. The spectrum and the synthesis condition. Let $S = S(\mathbf{R})$ denote the Schwartz space and let, for a distribution $T \in S'$, $\langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle$, $\varphi \in S$, denote the Fourier transform. Then a basic notion of harmonic analysis, the spectrum of T, is defined by $S_p(T) := \operatorname{supp}(\hat{T}^*)$ (cf. [9], p. 27), where $\langle T^*, \varphi \rangle := \langle T, \varphi^* \rangle$. There are several other definitions of the spectrum of $\mathscr{L}^q$-functions, which turn out to be equivalent to this one. Since S for $p \in [1, \infty]$ can be considered as a subspace of $\mathscr{L}^p_{w_\infty} = \bigcap_{N=1}^{\infty} \mathscr{L}^p_{w_N}$, it follows that

$$\mathscr{L}^q_{w_\infty^-} = \bigcup_{N=1}^{\infty} \mathscr{L}^q_{w_N^-}$$

is a subspace of S'. Therefore, for $f \in \mathscr{L}^q_{w_{\bar{\infty}}}$ the spectrum is defined.

THEOREM 5. *Let for $f \in \mathscr{L}^1_{w_N}$ and $1 \leqslant q < \infty$ the uniqueness condition* (U_q) *be satisfied*:

$$(\mathrm{U}_q)\colon \big(\forall g \in \mathscr{L}^q_{w_N} \text{ with } \mathrm{Sp}(g) \subset Zf \Rightarrow g = 0\big).$$

Then $\mathscr{F} = \{L_y f\colon y \in \boldsymbol{R}^1\}$ is $\mathscr{L}^q_{w_{\bar{N}}}$-complete.

Proof. For the proof we need the following representation.

LEMMA 5. *If $g \in \mathscr{L}^q_{w_{\bar{N}}}$, then*

$$\begin{aligned} \mathrm{Sp}(g) &= \{t \in \boldsymbol{R}^1\colon \forall \text{ open } U = U(t),\ \exists f \in \mathscr{L}^1_{w_N} \\ &\qquad \text{with } \mathrm{supp}(\hat{f}) \subset U(t) \text{ and } \langle \hat{g}, \hat{f} \rangle \neq 0\} \\ &= \bigcap \{K \subset \boldsymbol{R}^1 \text{ closed}\colon f * g^*(0) = \langle \hat{g}, \hat{f} \rangle = 0 \text{ for all } f \in \mathscr{L}^1_{w_N} \\ &\qquad \text{with } \mathrm{supp}(\hat{f}) \subset K^c\}. \end{aligned}$$

Proof. The inclusion "$\subset$" follows from a well-known construction method and the fact that $f \in \mathscr{L}^1_{w_N}$ iff $\hat{f}$ is N-fold differentiable.

($\supset$) Suppose that for x there exists $f \in \mathscr{L}^1_{w_N}$ and an open neighbourhood $U = U(x)$ with $\mathrm{supp}(\hat{f}) \subset U$; w.l.g. $\mathrm{supp}(f)$, $\mathrm{supp}(\hat{f})$ are compact and $f \in \mathscr{L}^\infty$. Therefore, $f \in \mathscr{L}^2$ and, by Fourier's inversion theorem,

$$f(x) = \frac{1}{2\pi} \int e^{itx} \hat{f}(t)\, dt.$$

Since $\mathrm{supp}(\hat{f}) \subset U(x)$ is compact, there exists a relatively compact neighbourhood $\tilde{U}(x)$ and $\varepsilon > 0$ with $\mathrm{supp}(\hat{f}) \subset \tilde{U}(x)$ and $\tilde{U}(x) + U_\varepsilon(0) \subset U(x)$, $U_\varepsilon(0)$ denoting the ε-neighbourhood of zero. For a sequence $\varepsilon > \varepsilon_n \downarrow 0$ define a sequence $(h_{m_n})_{m \in N}$ by

$$\hat{h}_{0_n} := f, \qquad \hat{h}_{m_n} := (M_{\varepsilon_n/2m^2} \psi_1)^\wedge * \hat{h}_{(m-1)_n},$$

where $M_r f(x) = rf(rx)$ is the multiplication operator and

$$\psi_1(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

Then $\hat{h}_{m_n}$ has m-th derivatives and

$$\mathrm{supp}(\hat{h}_{m_n}) = \overline{\mathrm{supp}(\hat{h}_{(m-1)_n}) + U_{\varepsilon/2m^2}(0)}.$$

For $n \in N$ we define $\hat{h}_n := \lim_{m \to \infty} \hat{h}_{m_n}$. Then $\hat{h}_n$ is infinitely often differentiable and

$$\begin{aligned} \mathrm{supp}(\hat{h}_n) &\subset \mathrm{supp}(\hat{f}) + \bigoplus_{m=1}^{\infty} \overline{U_{\varepsilon_n/2m^2}(0)} = \mathrm{supp}(\hat{f}) + \overline{U_{\Sigma \varepsilon_n/2m^2}(0)} \\ &= \mathrm{supp}(\hat{f}) + \overline{U_{\varepsilon_n}(0)} \subset U(x), \end{aligned}$$

where the summation in the subscript is over $m = 1, 2, \ldots$ Furthermore, $\lim \hat{h}_n = \hat{f}$ and $\exists C \in \boldsymbol{R}$ with $|\hat{h}_n(t)| < C$, $\forall t, n$. This implies, by dominated convergence,

$$\lim_{n\to\infty} h_n(x) = \frac{1}{2\pi} \lim_{n\to\infty} \int_{\operatorname{supp}(\hat{f})+U_\varepsilon(0)} \hat{h}_n(t) e^{itx}\, dt = \frac{1}{2\pi} \int \lim_{n\to\infty} \hat{h}_n(t) e^{itx}\, dt = f(x).$$

Also $|h_n| \leqslant C\lambda(\operatorname{supp}(\hat{f}) + U_\varepsilon(0)) < \infty$ and $\operatorname{supp}(h_n) \subset \operatorname{supp}(f)$, $n \in N$. Suppose that, $\forall n \in N$, $\langle \hat{h}_n, \hat{g} \rangle = \langle h_n, g \rangle = 0$; then we obtain

$$|f * g^*(0)| = \left|\int f(x) g(x)\, dx\right| \leqslant \overline{\lim} \int |f(x) - h_n(x)|\, g(x)\, dx + \underbrace{\overline{\lim} \left|\int h_n(x) g(x)\, dt\right|}_{=0}$$

$$\leqslant \overline{\lim}\, \|f - h_n\|_{1,w_N} \|g\|_{q,w_{\bar{N}}} = \|g\|_{q,w_{\bar{N}}} \Big(\overline{\lim} \int_{\operatorname{supp}(f)} |f(x) - h_n(x)|\, w_N(x)\, dx\Big)$$

$$\leqslant \|g\|_{q,w_{\bar{N}}} \int_{\operatorname{supp}(f)} \lim |f(x) - h_n(x)|\, w_N(x)\, dx = 0,$$

a contradiction. Therefore, there exists an infinitely often differentiable element $\hat{h}_n$ with $\operatorname{supp}(\hat{h}_n) \subset U(x)$ and $g^* * h_n(0) \neq 0$, implying that $x \in \operatorname{Sp}(g)$. ∎

The proof of Theorem 5 now follows from the following result which extends Proposition 1.4.1 of Benedetto [1].

THEOREM 6. *For* $g \in \mathscr{L}^q_{w_{\bar{N}}}$ *we have*

$$\operatorname{Sp}(g) = \{t \in \boldsymbol{R}^1 : \hat{f}(t) = 0,\ \forall f \in \mathscr{L}^1_{w_N} \text{ with } g^* * f = 0\}.$$

Proof. ($\supset$) If $t_0 \notin \operatorname{Sp}(g)$, then $\exists U = U(t_0) \subset \boldsymbol{R}$ open such that: $g^* * f = 0$, $\forall f \in \mathscr{L}^1_{w_N}$ with $\operatorname{supp}(\hat{f}) \subset U$. Furthermore, $\exists n \in N$: $\overline{U_{1/n}(t_0)} \subset U$. For $f_0(x) := e^{it_0x} M_{1/n} \psi_N(x)$,

$$\psi_N(x) := \big(((M_{1/N}\psi_1)^\wedge)^N(0)\big)^{-1} (M_{1/N}\psi_1(x))^N$$

holds, $f_0 \in \mathscr{L}^1_{w_N}$ and $\hat{f}_0 = 1 \neq 0$ imply

$$t_0 \notin \{t : \hat{f}(t) = 0,\ \forall f \in \mathscr{L}^1_{w_N} \text{ with } g^* * f = 0\}.$$

($\subset$) $\hat{\mathscr{I}}_N(\mathscr{M}_g) := \{\hat{f} \in \mathscr{A}^1_{w_N} : f * g^* = 0\}$ is a closed ideal in $\mathscr{A}^1_{w_N}$ with

$$Z(\hat{\mathscr{I}}_N(\mathscr{M}_g)) = \bigcap_{\hat{f} \in \hat{\mathscr{I}}(\mathscr{M}_g)} Z\hat{f} = \{t : \hat{f}(t) = 0,\ \forall f \in \mathscr{L}^1_{w_N} \text{ with } f * g^* = 0\}.$$

We prove that for all $h \in \mathscr{L}^1_{w_N}$ with $\operatorname{supp}(h) \subset \big(Z(\hat{\mathscr{I}}_N(\mathscr{M}_g))\big)^c$ it follows that $h \in \hat{\mathscr{I}}_N(\mathscr{M}_g)$.

Indeed, $\hat{\mathscr{I}}_N(\mathscr{M}_g)$ is closed w.r.t. $\|\ \|_{\mathscr{A}^1_{w_N}}$; so w.l.g. we suppose that $\operatorname{supp}(\hat{h})$ is compact. For all $t_0 \in \operatorname{supp}(\hat{h})$, $\exists \hat{k}_{t_0} \in \hat{\mathscr{I}}_N(\mathscr{M}_g)$, $\varepsilon_{t_0} > 0$ such that

$$\hat{k}_{t_0}(t) \neq 0,\ \forall t \in U_{\varepsilon_{t_0}}(t_0) \subset \big(Z(\hat{\mathscr{I}}_N(\mathscr{M}_g))\big)^c.$$

There exists $k^-_{t_0} \in \mathscr{L}^1_{w_N}$ with $\hat{k}^-_{t_0}(t) = (\hat{k}_{t_0}(t))^{-1}$ for all $t \in \overline{U_{\varepsilon_{t_0}}(t_0)}$. If $h_{t_0} :=$

$:= k_{t_0} * k_{t_0}^{-} * h$, then $\hat{k}_{t_0} \in \hat{\mathscr{I}}_N(\mathscr{M}_g)$ implies $k_{t_0} * g^* = 0$ and also $h_{t_0} * g^* = 0$. This implies that $\hat{h}(t) = \hat{h}_{t_0}(t)$, $\forall t \in U_{\varepsilon_{t_0}}(t_0)$ for any $t_0 \in \operatorname{supp}(\hat{h})$. For $t_0 \notin \operatorname{supp}(\hat{h})$ we have $\hat{h}(t) = 0 \in \hat{\mathscr{I}}_N(\mathscr{M}_g)$ in a small neighbourhood of t_0. Therefore, $\hat{h}$ belongs locally to $\hat{\mathscr{I}}_N(\mathscr{M}_g)$. By the location lemma, therefore, $\hat{h} \in \hat{\mathscr{I}}_N(\mathscr{M}_g)$. With the closed set $K_0 := Z(\hat{\mathscr{I}}_N(\mathscr{M}_g))$ from Lemma 5 it follows that

$$\operatorname{Sp}(g) = \bigcap \{K\colon K \text{ closed}, f * g^* = 0, \forall f \in \mathscr{L}^1_{w_N} \text{ with } \operatorname{supp}(\hat{f}) \subset K^c\}$$
$$\subset K_0 = Z(\hat{\mathscr{I}}_N(\mathscr{M}_g)). \blacksquare$$

For $w = 1$, $1 \leqslant q \leqslant 2$, the Fourier transform of g^*, $g \in \mathscr{L}^q = \mathscr{L}^q_{\bar{w}}$, is well defined,

$$\hat{g}(t) = \lim_{R \to \infty} \int_{-R}^{R} e^{-itx} g(x)\,dx,$$

and the definition of the spectrum is equivalent to the usual definition.

THEOREM 7. *For* $g \in \mathscr{L}^q$, $1 \leqslant q \leqslant 2$, *we have* $\operatorname{Sp}(g) = \operatorname{supp}(\hat{g}^*)$.

Proof. ($\subset$) For $t_0 \notin \operatorname{supp}(\hat{g}^*)$, $\exists U_\varepsilon(t_0)$ such that $\hat{g}^*(t_0) = 0$, $\forall t \in \overline{U_\varepsilon(t_0)}$. Defining $h(x) := e^{it_0x} M_\varepsilon \psi_1(x)$, we obtain $\hat{h}(t_0) \neq 0$, $\operatorname{supp}(\hat{h}) = \overline{U_\varepsilon(t_0)}$, implying $\hat{g}^* \hat{h} = 0$, and so $g^* * h = 0$.

($\supset$) If $t_0 \notin \operatorname{Sp}(g)$, then, for some $\varepsilon > 0$, $\hat{g}^* \hat{h} = (g^* * h)^\wedge = 0$, $\forall h \in \mathscr{L}^1$ with $\operatorname{supp}(\hat{h}) \subset U_\varepsilon(t_0)$. For $h(x) = e^{it_0x} M_\varepsilon \psi_1(x)$ we have $\hat{h}(t_0) \neq 0$, $\operatorname{supp}(\hat{h}) \subset U_\varepsilon(t_0)$. Therefore, $\hat{g}^* \hat{h} = 0$ implies $\hat{g}^*(t_0) = 0$, $\forall t \in U_\varepsilon(t_0)$, and so $t_0 \notin \operatorname{supp}(\hat{g}^*)$. $\blacksquare$

For $q > 2$ or $w_N = (1+|x|)^N$, $g \in \mathscr{L}^q_{w_N}$ the limit $\lim \int_{-R}^{R} e^{-itx} g(x)dx$ does not exist generally. But with kernel functions h_σ satisfying $h_\sigma(x) \to 1$, $\sigma \to 0$, one can define the Fourier transform of the product $g^* h_\sigma(x)$. We use $h_\sigma(x) = e^{-\sigma|x|}$, $\sigma > 0$. Define for $g \in \mathscr{L}_{w_{\bar{\infty}}}$

$$U_g(\sigma, t, 0) := (g^* e^{-\sigma|\cdot|})^\wedge(t) = \int e^{itx} g(x) e^{-\sigma|x|}\,dx. \tag{12}$$

Then the following theorem extends results of Herz [12], Beurling [3] and Pollard [25] to the case of weight functions. We omit the somewhat lengthy proof.

THEOREM 8. *For* $g \in \mathscr{L}^q_{w_{\bar{N}}}$, $f \in \mathscr{L}^1_{w_N}$ *the following holds*:
1. *If* $\operatorname{Sp}(g) \subset K$, K *closed, then* $\lim_{\sigma \to 0} U_g(\sigma, t, 0) = 0$, $\forall t \notin K$.
2. *If* $K \subset (\operatorname{Sp}(g))^c$ *is compact, then* $U_g(\sigma, t, 0) \xrightarrow[\sigma \to 0]{} 0$ *uniformly on* K.
3. $\operatorname{Sp}(g) = \overline{\{t\colon \lim_{\sigma \to 0} U_g(\sigma, t, 0) \neq 0\}}$.

For the converse of Theorem 5 the following results hold:

THEOREM 9. (a) *For* $T \in (C_0(\boldsymbol{R}))'$, *the set of bounded Radon measures on* $\boldsymbol{R}$, $\varphi \in C_0(\boldsymbol{R})$, *the following synthesis condition holds*:

$$\operatorname{supp}(T) = \operatorname{Sp}(\hat{T}^*) \subset Z\varphi \Rightarrow \langle T, \varphi \rangle = 0.$$

(b) *If* $f \in \mathscr{L}^1$ *and if* $\mathscr{F} = \{L_y f\colon y \in \boldsymbol{R}^1\}$ *is* $\mathscr{L}^q$*-complete,* $1 \leqslant q \leqslant 2$, *then the condition* (U_q) *holds*:

$$\big(\forall g \in \mathscr{L}^q \text{ with } \operatorname{Sp}(g) \subset Z\hat{f} \Rightarrow g = 0\big).$$

Proof. (a) The proof can be given as for Theorem 1.3.2 of Benedetto [1], pp. 50–51.

(b) If $g \in \mathscr{L}^q$, $\operatorname{Sp}(g) \subset Z\hat{f}$, then $\hat{g}^* \in \mathscr{L}^p$. So (b) follows from (a).

A different argument is the following: $\operatorname{Sp}(g) = \operatorname{supp}(\hat{g}^*) \subset Z\hat{f} \Rightarrow \hat{g}^*\hat{f} = 0 \Rightarrow g^* * f = 0$, which by $\mathscr{L}^q$-completeness of $\mathscr{F}$ implies that $g^* = 0$, i.e. $g = 0$. ∎

For $q > 2$ the synthesis condition does not hold generally, but there are some results of Kinukawa [17]–[19], Herz [11] and Pollard [24].

THEOREM 10 (a) (cf. [19]). *If* $f \in \mathscr{L}^1$, $1 < p < 2$, *and* $|f(x)|^p \leqslant v(|x|)$ *for some* $v \in \mathscr{L}^1([0, \infty))$, $v\downarrow$, *then for* $g \in \mathscr{L}^q \cap \mathscr{L}^\infty$ *with* $\operatorname{Sp}(g) \subset Z\hat{f}$ *it follows that* $g^* * f = 0$ *(synthesis condition).*

(b) (Herz [11]). *If* $f \in \mathscr{L}^1$, $\hat{f} \in \mathscr{L}\mathrm{ip}(1)$, $g \in \mathscr{L}^q$, $q \geqslant 2$, $\operatorname{Sp}(g) \subset Z\hat{f}$, *then* $f * g^* = 0$.

(c) *If* $f \in \mathscr{L}^1$, $\hat{f} \in \mathscr{L}\mathrm{ip}(\varepsilon)$ *for some* $\varepsilon > 0$ *and* $\mathscr{F} = \{L_y f\colon y \in \boldsymbol{R}^1\}$ *is* $\mathscr{L}^q$*-complete,* $q \geqslant 2$, *then the condition* (U_q) *holds*: $\big(\forall g \in \mathscr{L}^\infty \cap \mathscr{L}^q$ *with* $\operatorname{Sp}(g) \subset Z\hat{f} \Rightarrow g = 0\big)$.

(d) (Pollard [24] for $\varepsilon = 1$). *If* $f \in \mathscr{L}^1$, $\varepsilon > 0$, $\int |f(x)|\,|x|^\varepsilon dx < \infty$ *and if* $\mathscr{F}$ *is* $\mathscr{L}^q$*-complete, then for* $g \in \mathscr{L}^q$ *with* $\operatorname{Sp}(g) \subset Z\hat{f}$ *it follows that* $g = 0$.

Remark. (d) follows from (c) by using the inequality:

$$|\hat{f}(t) - \hat{f}(t')| = \Big|\int (e^{-itx} - e^{-it'x}) f(x)\,dx\Big|$$

$$\leqslant |t - t'|^\varepsilon \int \frac{e^{-itx} - e^{-it'x}}{|tx - t'x|^\varepsilon} |x|^\varepsilon |f(x)|\,dx \leqslant M\,|t - t'|^\varepsilon \int |f(x)|\,|x|^\varepsilon dx$$

if $0 < \varepsilon \leqslant 1$, i.e., $\hat{f} \in \mathscr{L}\mathrm{ip}(\varepsilon)$. ∎

2.3. $\mathscr{L}^p_w$**-closedness /** $\mathscr{L}^q_{w^-}$**-completeness.** The first result extends a theorem of Beurling [4], who considered the case $w = 1$.

THEOREM 11. *If* $f \in \mathscr{L}^1_w$ *and* $\mathscr{F} = \{L_y f\colon y \in \boldsymbol{R}^1\}$ *is* $\mathscr{L}^q_{w^-}$*-complete,* $1 \leqslant q \leqslant \infty$, *then* $\mathscr{F}$ *is* $\mathscr{L}^{q'}_{w^-}$*-complete for* $q' \leqslant q$.

Proof. If $q' \leqslant q$ and $g \in \mathscr{L}^{q'}_{w^-}$, $g \not\equiv 0$ satisfies $g^* * f \equiv 0$, then define

$$h(x) := \int_{|x| < |y| < |x+1|} g^*(y)\,dy.$$

By Hölder's inequality it is not hard to show that $h \in \mathscr{L}^q_{w^-}$, considering the cases $q < \infty$ and $q = \infty$ separately.

If $q = \infty$, then

$$\operatorname{ess\,supp}_x |h(x)\,w^{-1}(x)| = \operatorname{ess\,supp}_x \Big| \int_{|y|<1} g^*(x+y)\,w^{-1}(x)\,dy \Big|$$
$$\leqslant \sup_x \Big(\int_{|y|<1} |g^*(x+y)|^{q'} w^{-q'}(x)\,dy \Big)^{1/q'}$$
$$\leqslant \big(\sup_{|y|<1} w(y) \big) \sup_x \Big(\int_{|y|<1} |g^*(x+y)|^{q'} \big(w(x)\,w(y)\big)^{-q'} dy \Big)^{1/q'}$$
$$\leqslant \sup_{|y|<1} w(y) \sup_x \Big(\int_{|y|<1} |g^*(x+y)|^{q'} \big(w(x+y)\big)^{-q'} dy \Big)^{1/q'}$$
$$\leqslant \big(\sup_{|y|<1} w(y) \big) \|g^*\|_{q',w^-} < \infty. \ \blacksquare$$

If $q < \infty$, then for $g^* \in \mathscr{L}^{q'}_{w^-}$ we have

$$\int |g^*(y)|^{q'} \big(w(y)\big)^{-q'} dy = \sum_{x=0}^{\infty} a_x, \qquad a_x := \int_{|x|<|y|<|x+1|} |g^*(y)|^{q'} \big(w(y)\big)^{-q'} dy.$$

Since $a_x \xrightarrow[x\to\infty]{} 0$, there exists a constant K such that for any $x \in \boldsymbol{R}_+$, $x > K$,

$$\int_{|x|<|y|<|x+y|} |g^*(y)|^{q'} \big(w(y)\big)^{-q'} dy \leqslant a_{[x]} + a_{[x+1]}.$$

This implies

$$\int_{|x|>K} |h(x)|^q \big(w(x)\big)^{-q} dx = \int_{|x|>K} \Big| \int_{|x|<|y|<|x+1|} g^*(y)\,dy \Big|^q \big(w(x)\big)^{-q} dx$$
$$= \int_{|x|>K} \Big| \int_{|y|<1} g^*(x+y)\,dy \Big|^q \big(w(x)\big)^{-q} dx$$
$$\leqslant \int_{|x|>K} \Big| \int_{|y|<1} |g^*(x+y)|^{q'} dy \Big|^{q/q'} \big(w(x)\big)^{-q} dx$$
$$\leqslant \int_{|x|>K} \left(\int_{|y|<1} \left(\frac{w(x+y)}{w(x)\,w(y)} \right)^{-q'} |g^*(x+y)|^{q'} dy \right)^{q/q'} \big(w(x)\big)^{-q} dx$$
$$\leqslant \sup_{|y|<1} \big(w(y)\big)^q \Big(\int_{|x|>K} \int_{|y|<1} \big(w(x+y)\big)^{-q'} |g^*(x+y)|^{q'} dy \Big)^{q/q'} dx$$
$$\leqslant \sup_{|y|<1} \big(w(y)\big)^q \int_{|x|>K} \int_{|y|<1} \big(w(x+y)\big)^{-q'} |g^*(x+y)|^{q'} dy\,dx$$
$$= \sup_{|y|<1} \big(w(y)\big)^q \int_{|y|<1} \int_{|x|>K} \big(w(x+y)\big)^{-q'} |g^*(x+y)|^{q'} dx\,dy$$
$$\leqslant \sup_{|y|<1} \big(w(y)\big)^q \|g\|_{q',w^-} < \infty.$$

A similar inequality holds for the integral on $\{|x| \leqslant K\}$.

Furthermore,

$$f*h(\theta)=\int f(x)\,h(\theta-x)\,dx=\int\Big(\int\limits_{|y|<1} g^*(y+\theta-x)\,dy\Big)\,dx$$
$$=\int\limits_{|y|<1}\int g^*(y+\theta-x)f(x)\,dx=\int\limits_{|y|<1} f*g^*(y+\theta)\,dy=0. \quad \blacksquare$$

The following generalized version of Theorem 11 holds:

THEOREM 12. *If $f_j\in\mathscr{L}^1_w$, $j\in J$, and if $\mathscr{F}=\{L_y f_j\colon y\in \boldsymbol{R}^1, j\in J\}$ is $\mathscr{L}^q_{w^-}$-complete, $1\leqslant q\leqslant\infty$, then $\mathscr{F}$ is $\mathscr{L}^{q'}_{w^-}$-complete for $q'\leqslant q$.* ■

Theorems 11 and 12 allow to introduce the *closure-exponent* $\gamma\geqslant 0$ defined by

$$\gamma:=\inf\{p\colon \mathscr{F} \text{ is } \mathscr{L}^p_w \text{ closed for all } p>\gamma\} \tag{13}$$
$$=\inf\{z\colon \mathscr{F} \text{ is } \mathscr{L}^q_{w^-} \text{ complete for all } q<(z-1)/z\}.$$

For $w=1$ cf. Beurling [4].

The following theorem is a modification of Wiener's original L^2-theorem; for a related version cf. [30].

THEOREM 13. (a) *If $q\leqslant 2$, $f\in\mathscr{L}^1$ and $\lambda(Z\hat{f})=0$, then $\mathscr{F}$ is $\mathscr{L}^q$-complete.*
(b) *If $q\geqslant 2$, $f\in\mathscr{L}^1\cap\mathscr{L}^p$, $p:=q/(q-1)$, $\mathscr{F}$ is $\mathscr{L}^q$-complete, then $\lambda(Z\hat{f})=0$.*

Proof. (a) If $g\in\mathscr{L}^q$, $g^* * f=0$, then $\hat{g}^*\hat{f}=(g^* * f)^\wedge=0$ and, therefore, $\hat{g}^*=0$ $[\lambda]$. Since $\hat{g}^*\in\mathscr{L}^q$, $\{\hat{g}^*=0\}$ is closed, i.e., $\hat{g}^*=0$, and so $g=0$.

(b) Since $\mathscr{F}$ is $\mathscr{L}^q$-complete, it is also $\mathscr{L}^p$-closed. For $h_m\in\mathscr{L}^p$ with $\hat{h}_m=1$ on $U_m(0)$ and compact support there exist $y_{m,1},\ldots,y_{m,n_m}$, $a_{m,1},\ldots,a_{m,n_m}\in\boldsymbol{C}$ with

$$\Big\|h_m-\sum a_{m,j}L_{y_{m,j}}f\Big\|_p \xrightarrow[n\to\infty]{} 0$$
$$\Rightarrow \Big\|\hat{h}_m-\sum_{j=1}^{n_m} a_{m,j}\exp(iy_{m,j})\hat{f}\Big\|_q\leqslant\Big\|\hat{h}_m-\sum a_{m,j}\exp(iy_{m,j})\hat{f}\Big\|_{\mathscr{A}^p}\xrightarrow[n\to\infty]{}0$$
$$\Rightarrow \lambda(Z\hat{f}\cap U_m(0))\leqslant\int\limits_{Z\hat{f}}|\hat{h}_m(t)|^q\,dt$$
$$\leqslant\int\limits_{Z\hat{f}}\Big|\hat{h}_m(t)-\sum a_{m,j}\exp(ity_{m,j})\Big|^q\hat{f}(t)\,dt$$
$$+\int\limits_{Z\hat{f}}\Big|\sum a_{m,j}\exp(ity_{m,j})\hat{f}(t)\Big|^q dt\to 0 \quad \text{for } n\to\infty$$
$$\Rightarrow \lambda(Z\hat{f})=0. \quad\blacksquare$$

Remark. Using Theorem 12 one can reduce Theorem 13 to the well-known case $q=2$, i.e.

(a) $\mathscr{F}$ is $\mathscr{L}^q$-complete, $q\geqslant 2\Rightarrow\mathscr{F}$ is $\mathscr{L}^2$-complete $\Leftrightarrow\lambda(Z\hat{f})=0$.
(b) If $q\leqslant 2$, $\lambda(Z\hat{f})=0\Rightarrow\mathscr{F}$ is $\mathscr{L}^2$-complete $\Rightarrow\mathscr{F}$ is $\mathscr{L}^q$-complete.

The following result is well known.

THEOREM 14. *If $f \in \mathscr{L}^1$, then $\mathscr{F}$ is $\mathscr{L}^1$-complete* $\Leftrightarrow (Z\hat{f})^\circ = \emptyset$.

Proof. ($\Rightarrow$) If $(Zf)^\circ \neq \emptyset$, then $U_\varepsilon(x_0) \subset Z\hat{f}$ for some $\varepsilon > 0$ and x_0. We can construct $h \in \mathscr{L}^1$ with $\hat{h} \not\equiv 0$, $\operatorname{supp}(\hat{h}) \subset U_\varepsilon(x_0)$ (e.g., $h = \exp[ix_0(\cdot)]\, M_{\varepsilon/2}\psi_1$). This implies that $(f * h(t))^\wedge = \hat{f}(t)\hat{h}(t) = 0$, a contradiction.

($\Leftarrow$) If $(Z\hat{f})^\circ = \emptyset$, then $(Z\hat{f})^c$ is dense in $\boldsymbol{R}^1$. If $h \in \mathscr{L}^1$, $f * h = 0$, then $\hat{f}\hat{h} \equiv 0$. Therefore, $\hat{h}(t) = 0$, $\forall t \in (Z\hat{f})^c$. Since $\hat{h} \in \mathscr{A}^1$ is continuous, we have $\hat{h} = 0$ and, therefore, $h = 0$. ∎

For $q \in (2, \infty)$ the following result is due to Herz [11] for $\alpha = 0$. For the proof it is shown that the uniqueness condition (U_q) holds.

THEOREM 15 (Herz [11], Theorem 4). *If $q \geqslant 2$, $f \in \mathscr{L}^1_{w_\alpha}$ and if, for each $E \subset Z\hat{f}$ compact,*

$$\lambda(E + U_\sigma(0)) = O(\sigma^{1-2/q+2\alpha}),$$

then $\mathscr{F}$ is $\mathscr{L}^q_{w_{\bar\alpha}}$-complete.

Proof. For the proof it is enough to establish the uniqueness condition for elements g with compact spectrum.

LEMMA 6. *Let $f \in \mathscr{L}^1_{w_\alpha}$; if for all $g \in \mathscr{L}^q_{w_{\bar\alpha}}$ with* $\operatorname{Sp}(g)$ *compact,* $\operatorname{Sp}(g) \subset Z\hat{f}$ *implies $g = 0$, then (U_q) holds.*

Proof. Define, for $g \in \mathscr{L}^q_{w_{\bar\alpha}}$ with $\operatorname{Sp}(g) \subset Z\hat{f}$, $g_n := M_n\psi_{[\alpha/2+1]} * g$; then $g_n \in \mathscr{L}^q_{w_{\bar\alpha}}$ and $\operatorname{Sp}(g_n) \subset \operatorname{Supp}\big((M_n\psi_{[\alpha/2+1]})^\wedge \hat{g}\big) \subset \operatorname{Supp}(\hat{g})$ is compact, i.e., $\operatorname{Sp}(g_n) \subset \operatorname{Sp}(g) \subset Z\hat{f}$. Therefore, $g_n = 0$, $\forall n$. Since $g_n \to g$, also $g \equiv 0$. ∎

Now for $g \in \mathscr{L}^q_{w_{\bar\alpha}}$ with $\operatorname{Sp}(g) \subset Z\hat{f}$ compact the following holds:

$$\int |I_{g\psi_\alpha}(\sigma, t, 0)|^2\, dt = O(\sigma^{2/q-1-2\alpha}),$$

$$\text{where } I_g(\sigma, t, y) := \int e^{-itx} e^{-\sigma|x+y|} g(x)\, dx.$$

Indeed,

$$\begin{aligned}
\int |I_{g\psi_\alpha}(\sigma, t, 0)|^2\, dt &= \int \Big|\int e^{-itx}\, 2\sigma^{-1} M_{\sigma/2}\psi_\alpha(x)\, g(x)\, dx\Big|^2 dt \\
&= 2\pi \int 2\sigma^{-1} |M_{\sigma/2}\psi_\alpha(x)\, g(x)|^2\, dt \\
&\leqslant C2\pi\, \|g\|_{q,w_{\bar\alpha}} \big(\|2\sigma^{-1} M_{\sigma/2}\psi_\alpha\|_{w_\alpha, 2q/(q-2)}\big)^2 \\
&= C2\pi\, \|g\|_{q,w_{\bar\alpha}} \Big(\int |\psi_\alpha(x)\, w_\alpha(2x/\sigma)|^{2q/(q-2)} (2/\sigma)\, dx\Big)^{1-2/q} \\
&\leqslant C2\pi\, \|g\|_{q,w_\alpha} (2/\sigma)^{1-2/q}\, C'(2/\sigma)^{2\alpha} \\
&\quad \times \Big(\int |\psi_\alpha(x)\, w_\alpha(x)|^{2q/(q-2)}\, dx\Big)^{1-2/q} = O(\sigma^{2/q-1-2\alpha}).
\end{aligned}$$

By the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\int |I_{g\psi_\alpha}(\sigma, t, 0)|\, dt &\leqslant \Big(\int |I_{g\psi_\alpha}(\sigma, t, 0)|^2\, dt\Big)^{1/2} \lambda\big(\operatorname{Sp}(gM_{\sigma/2}(\psi_\alpha))\big)^{1/2} \\
&\leqslant O(\sigma^{2/q-1-2\alpha})\, \lambda\big(\operatorname{Sp}(g + U_{\sigma/2}(0))\big)^{1/2} = o(1)
\end{aligned}$$

by our assumption. For the first inequality we use the relation

$$\operatorname{Sp}(gh) = \overline{\operatorname{Sp}(g) - \operatorname{Sp}(h)},$$

valid for $g, h \in \mathscr{L}^{\infty}_{w_{\bar{\infty}}}$. Now we obtain

$$|g(x)| = \lim_{\sigma \to 0} |(2/\sigma) M_{\sigma/2} \psi_\alpha(x) g(x)| = \lim_{\sigma \to 0} (1/2\pi) |(2/\sigma)(M_{\sigma/2} \psi_\alpha g)^\wedge (x)|$$

$$= \lim_{\sigma \to 0} (1/2\pi) \left|\int e^{itx} I_{g\psi_\alpha}(\sigma, t, 0) dt\right| \leqslant \lim_{\sigma \to 0} (1/2\pi) \int |I_{g\psi_\alpha}(\sigma, t, 0)| \, dt = 0. \blacksquare$$

As a corollary we obtain

COROLLARY 1. *If* $f \in \mathscr{L}^1_{w_\alpha}$ *and* $Z\hat{f}$ *is countable discrete, then* $\mathscr{F}$ *is* $\mathscr{L}^q_{w_{\bar{\alpha}}}$*-complete for all* $q < 1/\alpha$.

Proof. Each compact set $E \subset Z\hat{f}$ is finite. Therefore,

$$\lambda(E + U_h(0)) = O(h) = o(h^{1-2/q+2\alpha}) = o(1) \quad \text{for } q < 1/\alpha. \blacksquare$$

In particular, if $\alpha = 0$, $f \in \mathscr{L}^1$, and if $Z\hat{f}$ is countable discrete, then $\mathscr{F}$ is $\mathscr{L}^q$-complete for all $q < \infty$.

The following result is due to Newman [21].

THEOREM 16. *Let* $p \in (1, 2]$, $\alpha := 2(p-1)/p =: 2/q$, $f \in \mathscr{L}^1$. *If* $Z\hat{f}$ *has the strong* α*-measure* 0, *then* $\mathscr{F}$ *is* $\mathscr{L}^q$*-complete.*

Remind that $S \subset \boldsymbol{R}^1$ closed has *strong* α*-measure zero* if

$$\varliminf_{n \to \infty} n^{1/\alpha - 1} r_{n_m} = 0 \quad \text{for all } m,$$

where r_{n_m} is the Lebesgue measure of the complement in $[-m, m]$ of the union of the n largest intervals in $S \cap [-m, m]$.

The proof of Theorem 16 can be given in the following way:

1. For all $h \in \mathscr{L}^1 \cap \mathscr{L}^p$ there exists $k_i \in \mathscr{L}^p$, $\hat{k}_i \in \mathscr{L}\mathrm{ip}(1)$, such that $\|k_i * h - h\|_p \to 0$ and $\hat{k}_i(t) = 0$ for all t with $\hat{h}(t) = 0$, $i \geqslant i_0$.
The construction uses the condition on the strong α-measure of $Z\hat{f}$.

2. Step 1 allows us to apply Theorem 10 (b) of Herz.

Beurling proved the following result in terms of the Hausdorff dimension of $Z\hat{f}$. Our proof is based on the idea of Beurling but makes some arguments a bit more explicit.

THEOREM 17 (Beurling [4]). (a) *If* $q \geqslant 2$ *and* $\dim(Z\hat{f}) < 2/q$, *then* $\mathscr{F}$ *is* $\mathscr{L}^q$*-complete.*
(b) *If* $Z\hat{f}$ *is countable, then* $\dim(Z\hat{f}) = 0$.

Proof. (a) If $g \in \mathscr{L}^q$, $g \not\equiv 0$ satisfies $f * g^* = 0$, then $\operatorname{Sp}(g) \subset Z\hat{f}$, and

$$\alpha := \dim(Z\hat{f}) < 2/q \Rightarrow p > 2/(2-\alpha) \Rightarrow \exists \beta\colon p > 2/(2-\beta) > 2/(2-\alpha).$$

By an approximation argument used before w.l.g. $\mathrm{Sp}(g)$ is compact and $g \in \mathscr{L}^\infty$.

If $q = 2$, then

$$\int |g(x)|^2 |x|^{\beta-1} dx \leqslant \int_{|x|<1} \|g\|_\infty |x|^{\beta-1} dx + \int_{|x|\geqslant 1} |g(x)|^2 dx < \infty.$$

If $q > 2$, then

$$\int |g(x)|^2 |x|^{\beta-1} dx \leqslant \int_{|x|<1} (\|g\|_\infty)^2 |x|^{\beta-1} dx + \int_{|x|\geqslant 1} |x|^{\beta-1} |g(x)|^2 dx$$

$$\leqslant K\Big(\int_{|x|\geqslant 1} |g(x)|^{q/2} dx\Big)^{2/q}\Big(\int_{|x|\geqslant 1} |x|^{(q\beta-q)/(q-2)} dx\Big)^{(q-2)/q}$$

$$\leqslant K + \|g\|_q^2 \Big(\int_{|x|\geqslant 1} |x|^r dx\Big)^{(q-2)/q} < \infty, \quad r := \frac{q-q\beta}{q-2} > 1.$$

Therefore, with $\phi_{1-\beta}(x) := |x|^{\beta-1}$ the spectral measure $S_{\phi_{1-\beta}}(\mathrm{Sp}(g))$ is positive, and $S_{\phi_{1-\beta}}(\mathrm{Sp}(g)) = K_{\phi_\beta}(\mathrm{Sp}(g)) > 0$ (cf. [3], théorème 1), where the capacity K_ϕ is defined for $A \subset \boldsymbol{R}$ by

$$K_\phi(A) = \Big(\inf\{\textstyle\int\int \phi(x-y)\, d\mu(x)\, d\mu(y):$$

$$\mu \text{ a probability measure with } \mu(A) = 1\}\Big)^{-1}.$$

So there exists a probability measure $\mu \in M^1(\mathrm{Sp}(g))$ with

$$\int\int |x-y|^{-\beta} d\mu(x)\, d\mu(y) > 0.$$

This implies the existence of $E \subset \mathrm{Sp}(g)$ such that $\dim(E) > \alpha$. For the proof choose $E \subset \mathrm{Sp}(g)$ and $K \in \boldsymbol{N}$ with $\mu(E) > 0$ and

$$\int_E |x-y|^{-\beta} d\mu(y) < K < \infty \qquad \text{for all } x \in E.$$

Let μ_E be the restriction on E, $K' := K^{-1}\mu_E(E)$ and $\bigcup U_\sigma(x_i)$ a covering of E with balls of radius $\sigma_i < \sigma$. Then

$$|\sigma_i|^{-\beta}\mu_E(U_{\sigma_i}(x_i)) = |\sigma_i|^{-\beta}\mu_E(U_{\sigma_i}(x_i) \cap E) \leqslant \int_{E\cap U_{\sigma_i}(x_i)} |x_i - y|^{-\beta} d\mu(y)$$

$$\leqslant \int_E |x_i - y|^{-\beta} d\mu(y) < K,$$

implying that

$$0 < K' = K^{-1}\mu_E(E) \leqslant K^{-1}\sum_i \mu_E(U_{\sigma_i}(x_i)) \leqslant \sum |\sigma_i|^\beta.$$

Therefore, $\dim(E) \geqslant \beta$ and a contradiction follows from the inequalities

$$\dim(Z\hat{f}) \geqslant \dim(\mathrm{Sp}(g)) \geqslant \dim(E) \geqslant \beta > \alpha. \tag{14}$$

(b) is well known. ■

Remarks. (a) As a consequence of the results of this section we have the following classes of $\mathscr{L}^1$-functions:

1. $Z\hat{f} = \emptyset \Leftrightarrow \mathscr{F}$ is $\mathscr{L}^q$-complete for all $q \in [1, \infty]$;
2. $Z\hat{f}$ is countable $\Rightarrow \mathscr{F}$ is $\mathscr{L}^q$-complete for all $q < \infty$ (but not $q = \infty$);
3. $\lambda(Z\hat{f}) = 0 \Rightarrow \mathscr{F}$ is $\mathscr{L}^2$-complete;
4. $(Z\hat{f})^\circ = \emptyset \Rightarrow \mathscr{F}$ is $\mathscr{L}^1$-complete;
5. $(Z\hat{f})^\circ \neq \emptyset \Rightarrow \mathscr{F}$ is not $\mathscr{L}^q$-complete for any q;
6. there are some further specific criteria for $\mathscr{L}^q$-completeness in terms of the "dimension" of $Z\hat{f}$.

By a result of Pollard [25] and Sasvári [29] for any $A \subset \boldsymbol{R}^1$ closed there exists $f \in \mathscr{L}^1$ with $Z\hat{f} = A$. If $0 \notin A$, then f can be chosen as a probability density. If A is symmetric, $0 \notin A$, then f can be chosen in $\mathscr{L}^1_{w_\infty}$.

(b) Segal [30] explicitly constructs $f \in \mathscr{L}^1 \cup \mathscr{L}^p$ for any $p \in (1, 2)$ with $\lambda(Z\hat{f}) = 0$ but $\mathscr{F}$ *not* $\mathscr{L}^q$-complete, $1/p + 1/q = 1$, i.e., $\mathscr{F}$ is $\mathscr{L}^2$-complete but not $\mathscr{L}^q$-complete. $Z\hat{f}$ is chosen as a Cantor set (cf. also [28] and [21] for an explicit construction).

(c) With $A \in \mathscr{B}^1$ closed, $\lambda^1(A) > 0$, but A nowhere dense, we find $f \in \mathscr{L}^1$ with $Z\hat{f} = A$ such that $\mathscr{F}$ is $\mathscr{L}^1$-complete but $\mathscr{F}$ is not $\mathscr{L}^2$-complete. ∎

3. $\mathscr{L}^q(\mathscr{F})$-COMPLETENESS

We now consider the question of $\mathscr{L}^q(\mathscr{F})$-completeness of $\mathscr{F}$, i.e.

(15) $\forall g$ with $\int |g(x)|^q L_y f(x)\, dx < \infty$, $\forall y$ and $f * g^* = 0$ implies $g = 0$.

In contrast to (15) the completeness condition investigated so far was concerned instead $\mathscr{L}^q$ or $\mathscr{L}^q_{w^-}$ with weight functions w. Since $\mathscr{L}^\infty = \mathscr{L}^\infty(\mathscr{F}) \subset \mathscr{L}^q(\mathscr{F})$ for all $q \in [1, \infty]$, the condition $Z\hat{f} = \emptyset$ is necessary for $\mathscr{L}^q(\mathscr{F})$-completeness. The question now is to find additional conditions on f to ensure that $Z\hat{f} = \emptyset$ is sufficient for $\mathscr{L}^q(\mathscr{F})$-completeness. Define, for any h: $\boldsymbol{R}^1 \to \boldsymbol{R}^1$ measurable as in (6),

$$\mathscr{L}^q_h := \{g: \boldsymbol{R}^1 \to \boldsymbol{R}^1;\ \|gh\|_q < \infty\}. \tag{16}$$

Lemma 7. *For h, l: $\boldsymbol{R}^1 \to \boldsymbol{R}^1$, $c > 0$, the following holds*:
(i) $|h(x)| \leqslant c|l(x)|$, $\forall x \Rightarrow \mathscr{L}^q_l \subset \mathscr{L}^q_h$, $\forall q \in [1, \infty]$.
(ii) $hl^{-1} \in \mathscr{L}^p$, $1/p + 1/q = 1 \Rightarrow \mathscr{L}^q_l \subset \mathscr{L}^1_h$.

Proof. (i) is trivial.
(ii) If $g \in \mathscr{L}^q_l$, then by Hölder's inequality we obtain

$$\|g\|_{1,h} = \int |g(x)\, l(x)\, l^{-1}(x)\, h(x)|\, dx \leqslant \|gl\|_q \, \|hl^{-1}\|_p < \infty. \quad ∎$$

Lemma 8. *If w is a weight function, $q, q' \in [1, \infty]$. Then*:

$$\mathscr{L}^q_w \subset \mathscr{L}^{q'}_{f^{1/q'}} \Leftrightarrow \mathscr{L}^q_{w^-} \subset \mathscr{L}^{q'}(\mathscr{F}).$$

Proof. ($\Leftarrow$) follows from $\mathscr{L}^{q'}(\mathscr{F}) \subset \mathscr{L}^{q'}_{f^{1/q'}}$.

($\Rightarrow$) If $g \in \mathscr{L}^q_{w^-}$, then $w(x)w(y) \geqslant w(x+y)$ implies

$$\int |g(x+y)|^q (w(x))^{-q} dx \leqslant \int |g(x+y)|^q (w(x+y))^{-q} w(y)^q dx$$
$$= \left(\int |g(x)|^q w(x)^{-q} dx\right) w(y)^q < \infty \Rightarrow L_y g \in \mathscr{L}^q_{w^-} \Rightarrow \int |g(x)|^{q'} f(x-y)\, dx$$
$$= \int |g(x+y)|^{q'} f(x)\, dx < \infty \Rightarrow g \in \mathscr{L}^{q'}(\mathscr{F}). \quad \blacksquare$$

We obtain the following necessary conditions for $\mathscr{L}^q(\mathscr{F})$-completeness.

THEOREM 18. *For any weight function* w *the following holds*:

(i) *If* $|f|^{1/q} \leqslant Cw^{-1}$ *and if* $\mathscr{F}$ *is* $\mathscr{L}^q(\mathscr{F})$*-complete, then* $\mathscr{F}$ *is* $\mathscr{L}^q_{w^-}$*-complete.*

(ii) *If* $(fw)^{-1} \in \mathscr{L}^p$, *then* $\mathscr{L}^1(\mathscr{F})$*-completeness of* $\mathscr{F}$ *implies* $\mathscr{L}^q_{w^-}$*-completeness.*

Proof. (i) By Lemma 7, $\mathscr{L}^q_{w^-} \subset \mathscr{L}^q_{f^{1/q}}$; so, by Lemma 8, $\mathscr{L}^q_{w^-} \subset \mathscr{L}^q(\mathscr{F})$. Therefore, $\mathscr{L}^q(\mathscr{F})$-completeness of $\mathscr{F}$ implies $\mathscr{L}^q_{w^-}$-completeness.

(ii) By Lemma 7, $\mathscr{L}^q_{w^-} \subset \mathscr{L}^1_f$, and so, by Lemma 8, $\mathscr{L}^q_{w^-} \subset \mathscr{L}^1(\mathscr{F})$. So $\mathscr{L}^q_{w^-}$-completeness is a consequence of $\mathscr{L}^1(\mathscr{F})$-completeness. $\blacksquare$

THEOREM 19. (i) *If* $\mathscr{F}$ *is* $\mathscr{L}^q_{w^-}$*-complete and* $c|f|^{1/q} \geqslant w^{-1}$, *then* $\mathscr{F}$ *is* $\mathscr{L}^q(\mathscr{F})$*-complete.*

(ii) *If* $\mathscr{F}$ *is* $\mathscr{L}^1_{w^-}$*-complete and* $(f^{1/q}w)^{-1} \in \mathscr{L}^p$, *then* $\mathscr{F}$ *is* $\mathscr{L}^q(\mathscr{F})$*-complete.*

Proof. (i) By Lemma 1 with $h := w^{-1}$ and $l := f$ we have $\mathscr{L}^q(\mathscr{F}) \subset \subset \mathscr{L}^q_{f^{1/q}} \subset \mathscr{L}^q_{w^-}$, which implies (i).

(ii) For $h := w^{-1}$ and $l = f^{1/q}$ from Lemmas 7 and 8 it follows that $h/l = (f^{1/q}w)^{-1} \in \mathscr{L}^p$, and so $\mathscr{L}^q(\mathscr{F}) \subset \mathscr{L}^q_{f^{1/q}} \subset \mathscr{L}^1_{w^-}$. Therefore, $\mathscr{L}^1_{w^-}$-completeness of $\mathscr{F}$ implies $\mathscr{L}^q(\mathscr{F})$-completeness. $\blacksquare$

COROLLARY 1. *If* $fw \in \mathscr{L}^1$ *and*

(a) $C|f|^{1/q} \geqslant w^{-1}$ *or*

(b) $f^{-1/q}w^{-1} \in \mathscr{L}^p$, $1/p + 1/q = 1$,

then: $\mathscr{F}$ *is* $\mathscr{L}^q(\mathscr{F})$*-complete* $\Leftrightarrow Z\hat{f} = \emptyset$.

Proof. We have

$$\begin{aligned} Z\hat{f} = \emptyset \Leftrightarrow \mathscr{F} &\text{ is } \mathscr{L}^\infty_{w^-}\text{-complete} \\ \Rightarrow \mathscr{F} &\text{ is } \mathscr{L}^q_{w^-}\text{-complete}, \ \forall q \in [1, \infty] \quad \text{(by Theorem 11)} \\ \Rightarrow \mathscr{F} &\text{ is } \mathscr{L}^q(\mathscr{F})\text{-complete} \quad \text{(by Theorem 19)} \\ \Rightarrow \mathscr{F} &\text{ is } \mathscr{L}^\infty\text{-complete} \Rightarrow Z\hat{f} = \emptyset. \quad \blacksquare \end{aligned}$$

COROLLARY 2. *If* $\mathscr{F} = \{L_y f_j \colon y \in \boldsymbol{R}^1, j \in J\}$ *and* $f_j w \in \mathscr{L}^1$ *for all* $j \in J$, *and if for some* $j_0 \in J$

(a) $|f_{j_0}|\,|w|^q \geq C > 0$ *or*
(b) $f_{j_0}^{-p/q} w^{-p} \in \mathscr{L}^1$,
then: $\mathscr{F}$ *is* $\mathscr{L}^q(\mathscr{F})$*-complete* $\Leftrightarrow \bigcap_{j\in J} Z\hat{f}_j = \emptyset$.

Proof. By the results of Section 2, $\mathscr{F}$ is $\mathscr{L}^q_{w^-}$-complete for all $q \geq 1$ if $\bigcap_{j\in J} Z\hat{f}_j = \emptyset$. If $\mathscr{L}^q_{f^{1/q}_{j_0}} \subset \mathscr{L}^q_{w^-}$, then

$$\mathscr{L}^q(\mathscr{F}) \subset \bigcap_{j\in J} \mathscr{L}^q_{f_j^{1/q}} \subset \mathscr{L}^q_{w^-}.$$

The same relation holds for $\mathscr{L}^q_{f^{1/q}_{j_0}} \subset \mathscr{L}^1_{w^-}$. This implies Corollary 2. ∎

4. EXAMPLES

4.1. Uniform distribution. For $r > 0$ let

$$f(x) = \begin{cases} 1/2r, & |x| \leq r, \\ 0, & |x| > r. \end{cases}$$

Then

$$\hat{f}(t) = (\sin tr)/tr,$$

i.e., $Z\hat{f}$ is countable. This implies that $\mathscr{F}$ is not $\mathscr{L}^\infty$-complete, $\mathscr{F}$ is $\mathscr{L}^q$-complete for $1 \leq q < \infty$. Therefore, $\mathscr{F}$ is not $\mathscr{L}^q(\mathscr{F})$-complete for any $q \in [1, \infty]$.

4.2. Laplace distribution. For $\sigma > 0$ let

$$f(x) = (2\sigma)^{-1} e^{-|x|/\sigma}.$$

Then

$$\hat{f}(t) = (1+\sigma^2 t^2)^{-1}, \quad \text{i.e.,} \quad Z\hat{f} = \emptyset.$$

Therefore, $\mathscr{F}$ is $\mathscr{L}^q$-complete, $1 \leq q < \infty$. For $1 < q < \infty$ define $w_q(x) = e^{|x|/q\sigma}$; then there exists $C = C_q$ with $|f|^{1/q} w_q \geq C$ and $w_q f \in \mathscr{L}^1$. By Theorem 19, $\mathscr{F}$ is $\mathscr{L}^q(\mathscr{F})$-complete. For the case $q = 1$ the $\mathscr{L}^q(\mathscr{F})$-completeness has been shown by a different method by Oosterhoff and Schriever [23].

4.3. Normal distribution. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}.$$

Then

$$\hat{f}(t) = e^{-\sigma^2 t^2/2}, \quad \text{i.e.,} \quad Z\hat{f} = \emptyset.$$

Therefore, $\mathscr{F}$ is $\mathscr{L}^q$-complete for all $q \in [1, \infty]$. By our method with weight functions we cannot decide the $\mathscr{L}^q(\mathscr{F})$-completeness, since f is rapidly decreasing. The $\mathscr{L}^q(\mathscr{F})$-completeness for $q \in [1, \infty]$ here follows from the uniqueness of Laplace transforms. Related completeness results can also be proved for other densities based on analyticity properties as in the generalized Müntz-Szász theorem (cf. [5], [16]).

4.4. Cauchy distribution. Let

$$f(x) = 1/\pi(1+x^2).$$

Then

$$\hat{f}(t) = e^{-\theta|t|}, \quad \text{i.e.,} \quad Z\hat{f} = \emptyset$$

and, therefore, $\mathscr{F}$ is $\mathscr{L}^q$-complete for $q \in [1, \infty]$. For $w_q(x) = (1+|x|)^{2/q}$ we have $|f|^{1/q} w_q \geqslant C_q > 0$, $f w_q \in \mathscr{L}^1$ for $q \in (2, \infty]$. This implies that $\mathscr{F}$ is $\mathscr{L}^q(\mathscr{F})$-complete for all $q > 2$.

4.5. Gamma distribution. For $\lambda, \theta > 0$, let

$$f(x) = \frac{\theta^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\theta x}, \qquad x \geqslant 0.$$

Then

$$\hat{f}(t) = (1+it/\theta)^{-\lambda}.$$

Since the complex zeros of $\hat{f}$ are $Z_C \hat{f} = \{-\theta/i\}$, $Z\hat{f} = \emptyset$ and $\mathscr{F}$ is $\mathscr{L}^q$-complete for $q \in [1, \infty]$. By our method we cannot decide the $\mathscr{L}^q(\mathscr{F})$-completeness.

4.6. Logarithmic decrease (Fourier transforms from [23]). Let

$$f(x) = \frac{1}{2\pi} \ln(1+1/x^2).$$

Then

$$\hat{f}(t) = \begin{cases} t^{-1}(1+e^{-|t|}), & t \neq 0, \\ 1, & t = 0, \end{cases}$$

So $Z\hat{f} = \emptyset$; i.e., $\mathscr{F}$ is $\mathscr{L}^q$-complete for $q \in [1, \infty]$. For $q > 2$, $\alpha_q = 2(q-1)/q$, and $w_q(x) = (1+|x|)^{2-\alpha_q}$ we have

$$f(x)\, w_q(x) = \underbrace{\frac{1}{2\pi} \ln(1+1/x^2)(1+|x|)^2}_{\leqslant C} \frac{1}{(1+|x|)^{\alpha_q}} \in \mathscr{L}^1 \quad \text{and} \quad f^{1/q}(x)\,|w_q(x)| \geqslant C.$$

Therefore, for $2 < q \leqslant \infty$, $\mathscr{F}$ is $\mathscr{L}^q(\mathscr{F})$-complete.

4.7. Exponential decrease. Let

$$f(x) = \frac{1}{2\sqrt{\pi}} |x|^{-1/2} e^{-|x|}.$$

Then

$$\hat{f}(t) = \frac{1}{\sqrt{2}}(1+t^2)^{-1}(1+\sqrt{1+t^2})^{1/2}, \qquad Z\hat{f} = \emptyset,$$

which implies $\mathcal{L}^q$-completeness for $q \in [1, \infty]$. For $q \in (1, \infty)$ define

$$w_q(x) := \exp\left[\frac{1+q}{2}|x|\right];$$

then

$$f(x)\, w_q(x) = \frac{1}{2\sqrt{\pi}}|x|^{-1/2} \exp\left[\tfrac{1}{2}(q-1)|x|\right] \in \mathcal{L}^1,$$

$$|f(x)^{-p/q}|w_q(x)|^{-p} = (2\sqrt{\pi})^{p/q}|x|^{p/2q}\left(\exp\left[\tfrac{1}{2}(1/q-1)|x|\right]\right)^p \in \mathcal{L}^1$$

and, consequently, $\mathcal{F}$ is $\mathcal{L}^q(\mathcal{F})$-complete.

4.8. Several densities.

(a) Let

$$f_1(x) = \frac{4}{\pi(2+x^4)}, \qquad f_2(x) = \frac{1}{2\pi}\left(\frac{1}{1+(1-x)^2} + \frac{1}{1+(1+x)^2}\right);$$

then

$$\hat{f}_1(t) = \sqrt{2}\sin(\pi/4 + |t|)\, e^{-|t|}, \qquad \hat{f}_2(t) = \cos(t)\, e^{-|t|},$$

which implies $Z\hat{f}_1 \cap Z\hat{f}_2 = \emptyset$. For $q > 2$ and $\alpha_q = 2/q$ and $w_{\alpha_q}(x) = (1+|x|)^{\alpha_q}$ we have

$$|f_1(x)\, w_{\alpha_q}(x)| \leqslant \frac{C_1}{1+|x|^{4-\alpha_q}} \in \mathcal{L}^1 \quad \text{and} \quad |f_2(x) w_{\alpha_q}(x)| \leqslant \frac{C_2}{(1+|x|)^{2-\alpha_q}} \in \mathcal{L}^1$$

and for $j_0 = 2$ we obtain

$$|f_2(x)|\, |w_{\alpha_q}(x)|^q = \frac{1}{2\pi}\left(\frac{1+|x|^2}{1+(1-x)^2} + \frac{1+|x|^2}{1+(1+x)^2}\right) \geqslant C$$

for some constants C_1, C_2, $C > 0$. By Corollary 2, $\mathcal{F} = \{L_y f_i\colon y \in \boldsymbol{R}^1,\ i = 1, 2\}$ is $\mathcal{L}^q(\mathcal{F})$-complete for $q > 2$.

(b) Let

$$f_1(x) = |x|^{1/2} e^{-|x|}, \qquad f_2(x) = \frac{(\sqrt{2})^3}{\sqrt{\pi}}|x|^{1/2} e^{-2|x|};$$

then

$$\hat{f}_1(t) = \frac{1}{(1+t^2)^{3/4}}\cos\left(\frac{3}{2}\arctan|t|\right), \qquad \hat{f}_2(t) = \frac{(\sqrt{2})^3}{(2+t^2)^{3/4}}\cos\left(\frac{3}{2}\arctan\left|\frac{t}{2}\right|\right).$$

Therefore $Z\hat{f}_1 \cap Z\hat{f}_2 = \emptyset$. For $w(x) := e^{2/3|x|}$ we have

$$w(x)f_1(x) = |x|^{1/2} e^{-1/3|x|} \in \mathscr{L}^1, \quad w(x)f_2(x) = |x|^{1/2} e^{-4/3|x|} \frac{(\sqrt{2})^3}{\sqrt{\pi}} \in \mathscr{L}^1.$$

For $q > 3/2$ we get $p/2q < 1$, $1/q - 2/3 < 0$, and for $j_0 = 2$ we obtain

$$w^{-1/p}(x) f_1^{-p/q}(x) = |x|^{-p/2q} \exp[(1/q - 2/3)p|x|] \in \mathscr{L}^1.$$

By Corollary 2, $\mathscr{F} = \{L_y f_i: y \in \boldsymbol{R}^1, i = 1, 2\}$ is $\mathscr{L}^q(\mathscr{F})$-complete for $q > 3/2$.

5. SOME FINAL REMARKS ON ESTIMATION THEORY

Some basic characterization results for UMV-estimators in location families are proved in Bondesson [6], which are based on the Tauberian conditions $g^* * f = 0 \Rightarrow \mathrm{Sp}(g) \subset Z\hat{f}$ (cf. Theorem 6 and [6], Lemma 2.1) as well as on some synthesis conditions. Bondesson considers also estimators satisfying some weight conditions. A basic synthesis result used by Bondesson is the following extension of a result due to Hörmander [14] for $N = 0$:

If $f \in \mathscr{L}^1_{w_{N+\alpha}}$, $g \in \mathscr{L}^\infty_{w_{\bar{N}}}$, $\alpha > 0$, and if $\mathrm{Sp}(g) \subset \bigcap_{k=0}^N Z(\hat{f}^{(k)})$, then $f * g^* = 0$.

Furthermore, Bondesson proves that for distributions with an entire analytic characteristic function a nonperiodic UMV-estimator in $\mathscr{L}^\infty_{w_{\bar{N}}}$ can exist only for normal distributions or Dirac distributions.

Bounded completeness in connection with the Wiener closure theorem was used by Ghosh and Singh [10]. The following result refines their Theorem 2.1.

THEOREM 20. *If the density* $f \in \mathscr{L}^1_{w_N}$ *for some* $N \in \boldsymbol{N}$, $Z\hat{f} = \emptyset$ *and*

$$D_{\delta,N^-} = \{g \in \mathscr{L}^\infty_{w_{\bar{N}}}: \mathrm{E}_\theta g = \theta,\ \theta \in \boldsymbol{R}^1\},$$

then

$$D_{\delta,N^-} = \begin{cases} \emptyset & \text{if } N = 0, \\ \{\mathrm{id} - \mathrm{E}_0 X\} & \text{if } N > 0. \end{cases}$$

Proof. If $g \in D_{\delta,N^-}$, then for $y, \theta \in \boldsymbol{R}^1$

$$\int (g(x+y) - g(x) - y) f(x-\theta)\, dx = \mathrm{E}_{\theta+y} g - \mathrm{E}_\theta g - y = 0.$$

Since $\mathscr{F} = \{L_\theta f: \theta \in \boldsymbol{R}^1\}$ is $\mathscr{L}^\infty_{w_{\bar{N}}}$-complete, we obtain $L_\theta g = g + \theta$, $\theta \in \boldsymbol{R}^1$ a.s. Therefore, $g(x) = x + k$ for all x and some $k \in \boldsymbol{R}^1$. But $g \in \mathscr{L}^\infty_{w_{\bar{N}}} \Leftrightarrow N > 0$; so we obtain one inclusion. The other inclusion is obvious. ∎

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Institut für Math. Statistik
Universität Münster
Einsteinstr. 62
D-4400 Münster, Germany

Received on 31.7.1990;
revised version on 27.7.1992